17/11/23 MATH2050A Tutorial

Announcements:

- HW5 due 20/11

<u>Recall</u>: Thm (Sequential Criterion): $f: A \subseteq \mathbb{R} \to \mathbb{R}$, $c$ is a cluster pt. of $A$. TFAE:

1) $\lim_{x \to c} f = L$
2) For every sequence $(x_n) \subseteq A$ that converges to $c$ s.t. $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to $L$.

<u>Thm</u> (Divergence): $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $c \in \mathbb{R}$ is a cluster point of $A$. Then $f$ does not have a limit at $c$ iff $\exists (x_n) \subseteq A$ s.t. $x_n \neq c$ $\forall n \in \mathbb{N}$, s.t. $x_n \to c$ but $(f(x_n))$ does not converge.

<u>Q1</u>: Show using $\varepsilon$-$\delta$ definition and by the sequential criteria the following limit: $\lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$.

$\left( \exists \varepsilon_0 > 0 \text{ s.t. } \forall \varepsilon < \varepsilon_0 \ \exists \delta > 0, \ldots \right)$

Pf: $\varepsilon-\delta$ def'n. Let $0<\varepsilon<2$ be given.

$$\left|\frac{x}{1+x}-\frac{1}{2}\right| = \left|\frac{x-1}{2(1+x)}\right|.$$

Note if $|x-1|<1 \Rightarrow 1+x>1$
$\Rightarrow \frac{1}{|x+1|}<1.$

$$\overset{|x-1|<1}{<} \frac{1}{2}|x-1| < |x-1|.$$

So if I take $\delta=\min\{\varepsilon, 1\}$

If $\varepsilon>1$, then taking $\delta=1$, we have

$$\left|\frac{x}{1+x}-\frac{1}{2}\right| \underset{\delta=1 \Rightarrow |x-1|=1.}{<} \frac{1}{2}|x-1| < |x-1| < 1 < \varepsilon$$

If $\varepsilon<1$, then taking $\delta=\varepsilon$, we have

$$\left|\frac{x}{1+x}-\frac{1}{2}\right| \underset{\delta=\varepsilon<1}{<} \frac{1}{2}|x-1| < |x-1| = \delta = \varepsilon.$$

Sequential Criterion: Let $(x_n) \subseteq A$, $x_n \neq 1$ for all $n$ be a sequence s.t. $x_n \to 1$.

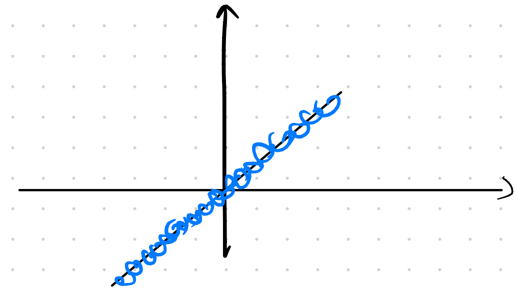
$\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $|x_n - 1| < \varepsilon$ for $n \geq N_\varepsilon$.

Then in particular, for $\varepsilon < 1$, $\exists N_\varepsilon$ s.t. $|x_n - 1| < \varepsilon \leq 1$, hence by ineq. above

$$\left| f(x_n) - \frac{1}{2} \right| = \left| \frac{x_n}{1+x_n} - \frac{1}{2} \right| = \left| \frac{x_n - 1}{2(x_n+1)} \right| < \frac{1}{2}|x_n - 1| < |x_n - 1| < \varepsilon.$$

showing the $\varepsilon$-N def'n of convergence for all $0 < \varepsilon < 1$.

Q2: $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\begin{cases} x & \text{if } x\in\mathbb{Q} \\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$



a) Show that f has a limit at 0.

b) Use a sequential argument to show that if $c\neq 0$, then f does not have a limit at c.

OPTIONALLY: Show f is nowhere continuous.

"pathological fn"

$g:\mathbb{R}\to\mathbb{R}$ by $g(x)=\begin{cases} 1 & x\in\mathbb{Q} \\ 0 & x\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$ is called "Dirichlet function"

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it is nowhere cts, not Riemann integrable, but it is Lebesgue integrable

Also: $g(x)=\lim_{k\to\infty}\left(\lim_{j\to\infty}(\cos(k!\pi x))^{2j}\right)$

Also: Weierstrass Function is an example of a function that is cts everywhere but is differentiable nowhere.

Pf: (a): Let $\varepsilon > 0$ be given.

$|f(x) - 0| = |f(x)| \leq |x|$. So take $\delta = \varepsilon$, then for $|x| < \delta$,

$|f(x) - 0| \leq |x| < \delta = \varepsilon$. So $\lim_{x \to 0} f = 0$.

b) Let $c \neq 0$. Since both $\mathbb{Q}$, $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$, for every $n$, we can find $q_n \in \mathbb{Q}$, $r_n \in \mathbb{R}\setminus\mathbb{Q}$ s.t. $|q_n - c| < \frac{1}{n}$

$|r_n - c| < \frac{1}{n}$

Then let $\varepsilon > 0$ be given. Then taking $N_\varepsilon \geq \frac{1}{\varepsilon}$, we have for all $n \geq N_\varepsilon$,

$|f(q_n) - c| = |q_n - c| < \frac{1}{n} < \varepsilon$. So $(f(q_n)) \to c$ as $n \to \infty$.

But for all $n \geq N_\varepsilon$,

$|f(r_n)| = 0$, so $(f(r_n)) \to 0$ as $n \to \infty$.

So $(f(x_n))$ does not converge where $x_n = \begin{cases} q_n & n \text{ odd} \\ r_n & n \text{ even} \end{cases}$